Introduction to Decision Sciences

Lecture 1

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Course Overview

I. Basic “mathematical literacy” and reasoning skills

- Logic: propositions, compound expressions, and quantifiers
- Proofs: direct, indirect, proof by cases, induction
- Sets, functions, and summations

II. Introduction/application to subjects of importance in decision sciences

- Number theory: divisibility, modular arithmetic, and primes
- Combinatorics: permutations, combinations, binomial coefficients
- Discrete probability
- Graphs and networks

True and False

Recall: the *positive integers* are the counting numbers $1, 2, 3, \ldots$

Consider the following statements (equations) involving the positive integers

$1 + 1 = 2$

$6 + 5 = 13$

$235 + 587 = 622$

$564 + 67 = 92$

Which equations are true, and which are false?

An Equation

The examples above are special cases of the equation

$$x + y = z$$

in which x, y, z are positive integers. Some terminology:

- If $x_0 + y_0 = z_0$ is *true*, we say x_0, y_0, z_0 is a *solution* of $x + y = z$
- Otherwise x_0, y_0, z_0 is not a solution

Fact: The equation $x + y = z$

- has a solution
- has infinitely many solutions

A Pythagorean Equation

Suppose again that x, y, z are positive integers and consider the equation

$$x^2 + y^2 = z^2$$

This new equation also has solutions, for example,

- $x = 3$, $y = 4$, $z = 5$ and $x = 5$, $y = 12$, $z = 13$

Fact: The equation $x^2 + y^2 = z^2$

- has a solution
- has infinitely many solutions

A Hard Problem

Suppose again that x, y, z are positive integers and consider the equation

$$x^3 + y^3 = z^3$$

It turns out that this equation has *no* solutions. More generally we have...

Fermat's Last Theorem: If $k \geq 3$ then the equation $x^k + y^k = z^k$ has no positive integer solutions.

- The theorem was stated by Pierre de Fermat around 1637.
- Andrew Wiles proved the theorem in 1995, over 350 years later!

One moral: A theorem that is easy to state may not be easy to prove.

Introduction to Logic

Logic is the formal language of mathematical reasoning.

- Used to express axioms and definitions
- Used to state and prove theorems

Basic Components

- Propositions
- Basic operations and conditional statements
- Compound propositions
- Quantifiers

Propositions

A *proposition* is a declarative statement or assertion that is either true (T) or false (F), but not both.

The *truth value* of a proposition is T if it is true, and F if it is false.

Propositions denoted by letters p, q, r, s, ..., called *propositional variables*

Examples

- p : Today is May 23rd
- $q : 3 + 2 = 5$
- $r : 4 \geq -5$
- s : UNC won the NCAA championship in 2017

Operations on Propositions

Basic Operations

- Negation (not)
- Conjunction (and)
- Dysjunction (or)
- Conditional (if-then)

Idea

- Operations combine simple propositions p and q to get new ones
- Truth value of the combination depends on the truth values of p and q
- Can keep track of truth values using truth tables

Negation

Negation of proposition p

- Denoted $\neg p$
- In words: “not p”.
- Truth value: $\neg p$ is T if p is F, and vice versa

Truth Table

Conjunction

Conjunction of propositions p and q

- Denoted $p \wedge q$
- In words: “p and q”.
- Truth value: $p \wedge q$ is T if p and q are T, and is false otherwise.

Truth Table

Disjunction

Disjunction of propositions p and q

- Denoted $p \vee q$
- In words: "p or q".
- Truth value: $p \vee q$ is T if at least one of p and q is T, and is F otherwise

Truth Table

Compound Proposition

A compound proposition is a combination of propositions using the basic operations. For example

- $\neg(p \wedge q)$
- $(\neg p) \vee (\neg q)$
- $\neg(p \vee q) \wedge q$

Conditional (If-Then) Statement

Implication: p implies q

- Denoted $p \to q$
- In words: “if p then q” (also “q if p”, “p sufficient for q”,...)
- Truth value: $p \to q$ is F only if p is T and q is F

Truth Table

Note: If p is F then $p \to q$ is T

Fact: $p \to q$ is logically equivalent to (has same truth values as) $\neg p \vee q$