STOR 655 Homework 13

1. Let (S, d) be a metric space, and let $N(S, \epsilon)$ be the covering number of S under the metric $d(.,.)$ at radius ϵ.

(a) What can you say about the limit of $N(S, \epsilon)$ as $\epsilon \to 0$? [Consider the case where S is finite and S is infinite.]

(b) Now let $S_0 \subseteq S$ be a subset of S. By definition, an ϵ-cover of S_0 contains of balls of radius ϵ centered at points in S_0, and $N(S_0, \epsilon)$ is the size of the smallest such cover. Consider instead general ϵ-covers of S_0 that are centered at points in S, so that centers need *not* be in S_0. Let $\tilde{N}(S_0, \epsilon)$ be the smallest such general cover. Find a simple relationship between $N(S_0, \epsilon)$ and $\tilde{N}(S_0, \epsilon)$.

2. Show that if $Y \sim \mathcal{N}(0, \sigma^2)$ then $\mathbb{E}\{|Y| I(|Y| > c)\} \leq \sigma \exp\{-c^2/2\sigma^2\}$

3. Show that $xy \leq 3x^2 + y^3/3$ for $x, y \geq 0$.

4. Show that if $d \geq 3$ then $\int_{\mathbb{R}^d} ||u||^{-2} \, e^{-||u||^2} = c \int_0^\infty e^{-r^2} r^{d-3} dr < \infty$.

5. Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim \mathcal{N}_n(\theta_i, 1)$. Suppose that we wish to simultaneously test the hypotheses $\mathrm{H}_{0,i} : \theta_i = 0$ vs. $\mathrm{H}_{1,i} : \theta_i \neq 0$ for $1 \leq i \leq n$. Consider a simple threshold test in which we reject $\mathrm{H}_{0,i}$ if $|X_i| > \tau$ and accept $\mathrm{H}_{0,i}$ otherwise. Using the asymptotic results on Gaussian extreme values, find a value of the threshold τ, depending on n, so that the family-wise error rate of the test under the global null $\theta_1 = \cdots = \theta_n = 0$ is (approximately) controlled at 5%.

6. Recall that the L_p-norm of a random variable X is defined by $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$. Establish Lyapunov's inequality: If $p \leq q$ then $\|X\|_p \leq \|X\|_q$. [Hint: Apply Hölder's inequality with an appropriate choice of conjugate exponents to $|X|^p \cdot 1$.]

7. Show that $\ln(1 - x) \leq -x - x^2/2$ for $0 \leq x < 1$