STOR 655 Homework 11

1. Recall that the convex hull of a set $A \subseteq \mathbb{R}^d$, denoted $\text{conv}(A)$, is the intersection of all convex sets C containing A. Show that $\text{conv}(A)$ is equal to the set of all convex combinations $\sum_{i=1}^k \alpha_i x_i$, where $k \geq 1$ is finite, $x_1, \ldots, x_k \in A$, and the coefficients α_i are non-negative and sum to one.

2. Identify the extreme points (if any) of the following convex sets.

   a. The hyperplane $H = \{x : x^t u = b\}$

   b. The halfspace $H_+ = \{x : x^t u > b\}$

   c. The closed ball $\overline{B}(x_0, r) = \{x : ||x - x_0|| \leq r\}$

3. Let $f : C \to \mathbb{R}$ be a strictly convex function defined on a convex set $C \subseteq \mathbb{R}^n$. Show that $\text{argmax}_{x \in C} f(x)$ is contained in the set of extreme points of C.

4. (Set sums and scaler products) Given sets $A, B \subseteq \mathbb{R}^d$ and a constant $\alpha \in \mathbb{R}$ define the set sum and set scaler product as follows:

$$A + B \;=\; \{x + y : x \in A \text{ and } y \in B\} \qquad \alpha A \;=\; \{\alpha x : x \in A\}$$

   a. (Optional) Show that if A is open then $A + B$ is open regardless of whether B is open.

   b. Show that if A and B are convex, then so is $A + B$.

   c. If A is convex is $A + B$ necessarily convex?

   d. Show by example that, in general, $2A \neq A + A$.

   d. Show that if A is convex then $\alpha A + \beta A = (\alpha + \beta)A$ for all $\alpha, \beta \geq 0$.

5. Let f be a convex function defined on a convex set C. Show that for each $\alpha \in \mathbb{R}$ the level set $L(\alpha) = \{x : f(x) \leq \alpha\}$ is convex.

6. Let $X \in \mathbb{R}$ be an integrable random variable with CDF $F(x)$, and for $0 < p < 1$ let $h_p(x, \theta) = p(x - \theta)_+ + (1 - p)(\theta - x)_+$.

   (a) Show that for each fixed p and x, $h_p(x, \theta)$ is a convex function of θ.

(b) Show that, under reasonable assumptions on F, the quantity $\mathbb{E}h_p(X, \theta)$ is minimized by the pth quantile $F^{-1}(p)$ of X. Clearly state any assumptions that you make.

(c) What does the result of part (b) tell you in the special case $p = 1/2$.

7. Let $A(t) = \{A_{i,j}(t) : 1 \leq i, j \leq n\}$ be a matrix whose entries are differentiable functions of a real number t. Define the entry-wise derivative

$$A'(t) \;=\; \{A'_{i,j}(t) : 1 \leq i, j \leq n\}$$

(a) Show that the entry-wise derivate obeys the usual product rule, that is,

$$[A(t)B(t)]' \;=\; A(t)B'(t) \,+\, A'(t)B(t)$$

(b) Suppose that for each t the matrix $A(t)$ is symmetric and non-negative definite. Then there exists a matrix function $A^{1/2}(t)$ such that $A(t) = A^{1/2}(t)A^{1/2}(t)$. Use part (a) to show that $A'(t) = 2A^{1/2}(t)\,(A^{1/2}(t))'$.

8. Show that $||x||_0 = \lim_{p \searrow 0} ||x||_p$ and that $||x||_\infty = \lim_{p \nearrow \infty} ||x||_p$.

9. Show that if $u, v \in \mathbb{R}^n$ are orthogonal then $||u||_2 + ||v||_2 \leq \sqrt{2}||u + v||_2$.

10. Establish the following facts about the Gaussian mean width $w(K)$ of a bounded set $K \subseteq \mathbb{R}^n$.

(a) If $K_1 \subseteq K_2$ then $w(K_1) \leq w(K_2)$

(b) $w(K) \geq 0$

(c) If $A \in \mathbb{R}^{n\times n}$ is orthogonal then $w(AK) = w(A)$

(d) For each $u \in \mathbb{R}^n$, $w(K + u) = w(K)$

(e) $w(K) = w(\mathrm{conv}(K))$

(f) $\sqrt{2/\pi}\,\mathrm{diam}(K) \leq w(K) \leq n^{1/2}\,\mathrm{diam}(K)$

(g) $w(K) \leq 2\,\mathbb{E}\sup_{x\in K}\langle x, V\rangle$ with $V \sim \mathcal{N}_n(0, I)$