STOR 655 Homework 9

1. Let $H_i(x_1^i) := \mathbb{E}[\, f(X_1^n) \,|\, X_1^i = x_1^i \,]$ be defined as in the proof of McDiarmid's inequality. Show carefully that

$$\sup_{u,u'} [\, H_i(x_1^{i-1}, u) \; - \; H_i(x_1^{i-1}, u') \,],$$

and note how your argument depends on the independence of $X_1, \ldots, X_n$.

2. Let $X_1, \ldots, X_n \in \mathcal{X}$ be i.i.d. and let $\mathcal{G}$ be a family of function $g : \mathcal{X} \to [-c, c]$. Define

$$f(x_1^n) \; = \; \sup_{g \in \mathcal{G}} \left| n^{-1} \sum_{i=1}^{n} g(x_i) \; - \; \mathbb{E} g(X) \right|$$

Find the difference coefficients $c_1, \ldots, c_n$ of f, and use these to establish concentration bounds for the random variable $f(X_1^n)$.

3. Let $X \sim \mathcal{N}_n(0, I)$ and $Y \sim \mathcal{N}_n(0, I)$ be independent multinormal random variables. For $0 \leq \theta \leq \pi/2$ define random vectors

$$X(\theta) \; = \; X \sin\theta \; + \; Y \cos\theta$$

$$\dot{X}(\theta) \; = \; X \cos\theta \; - \; Y \sin\theta$$

(a) Show that for each θ, $X(\theta)$ and $\dot{X}(\theta)$ have the same distribution as X.

(b) Show that for each θ, $X(\theta)$ and $\dot{X}(\theta)$ are independent.

4. *Concentration for norms of Gaussian random vectors.* Let $Y \sim \mathcal{N}_d(0, \Sigma)$ and consider the random variable $U = ||Y||$.

(a) Show that $U = F(X)$ where $X \sim \mathcal{N}_d(0, I)$ and $F(x) = ||\Sigma^{1/2} x||$

(b) Show that F Lipschitz with constant

$$L \; \leq \; \sup_{u \in \mathbb{R}^d} \frac{||\Sigma^{1/2} u||}{||u||}$$

(c) Find a bound on the right hand side of the inequality above involving the largest eigenvalue of Σ.

(d) Find a concentration inequality for U.

5. Let $X_1, \ldots, X_n \in \mathbb{R}^d$ be random vectors such that $||X_i|| \leq c_i/2$ with probability one, where $||u|| = (u^t u)^{1/2}$ is the ordinary Euclidean norm. Let $\alpha = (1/4) \sum_{i=1}^n c_i^2$.

(a) Show that $\mathbb{E}|| \sum_{i=1}^n X_i|| \leq \sqrt{\alpha}$.

(b) Use the bounded difference inequality and the inequality in part (a) to show that for all $t \geq \alpha$

$$P\left(|| \sum_{i=1}^n X_i|| > t \right) \;\leq\; \exp\left\{ \frac{(t - \sqrt{\alpha})^2}{2\alpha} \right\}$$

6. Let $a_1, \ldots, a_n$ be real numbers. Show that $n^{-1} \sum_{k=1}^n |a_k| \leq (n^{-1} \sum_{k=1}^n a_k^2)^{1/2}$.

7. Let $\Gamma(x)$ be the standard Gamma function, defined for $x > 0$. Show that if $Z \sim \mathcal{N}(0,1)$ then for each $p \geq 1$

$$\mathbb{E}|Z|^p \;=\; \frac{2^{p/2}}{\sqrt{\pi}} \Gamma((1+p)/2)$$

Deduce from this fact and Stirling's approximation that $||Z||_p := (\mathbb{E}|Z|^p)^{1/p} = O(p^{1/2})$.

8. Use Jensen's inequality to establish the Arithmetic-Geometric mean inequality: if $a_1, \ldots, a_n$ are positive constants, then

$$\frac{1}{n} \sum_{i=1}^n a_i \;\geq\; \left(\prod_{i=1}^n a_i \right)^{1/n}.$$

Hint: Logs will help.