STOR 655 Homework 8

1. Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with $\mathbb{E}X_i = p_i$. Let $S = X_1 + \cdots + X_n$ and let $\mu = \mathbb{E}S = \sum_{i=1}^n p_i$. Use Chernoff's bound and a MGF computation to show that for all $t > 0$

$$\mathbb{P}(S > t) \leq \exp\{t - \mu - t\log(t/\mu)\}$$

How does this bound compare to Hoeffding's inequality?

2. Recall that if $f : \mathcal{X} \to \mathbb{R}$ is a real-valued function then the argmax of f is the set of points in x at which f is maximized, $\arg\max_x f(x) = \{x : f(x) = \sup_{x' \in A} f(x')\}$. The argmin of f is similarly defined.

(a) Let $f : A \to \mathbb{R}$ be defined on a set $A \subseteq \mathbb{R}$ by $f(x) = x^2$. Identify the value of

$$\sup_{x \in A} f(x) \quad \text{and} \quad \arg\max_{x \in A} f(x)$$

in each of the following cases: $A = [-2, 2]$, $A = (-2, 2]$, $A = (-2, 2)$, and $A = (-3, 2]$.

(b) Let A be a bounded subset of $\mathbb{R}^d$. Identify the values of $\inf_x f(x), \sup_x f(x), \arg\min_x f(x)$, and $\arg\max_x f(x)$ for the following function:

$$f(x) = \inf_{y \in A} ||x - y||.$$

3. Let $S(x_1^n : \mathcal{A}) = |\{A \cap \{x_1, \ldots, x_n\} : A \in \mathcal{A}\}|$ be the shatter coefficient of a family $\mathcal{A} \subseteq 2^{\mathcal{X}}$. Show that for every sequence $x_1, \ldots, x_{m+n} \in \mathcal{X}$ we have the sub-multiplicative relation

$$S(x_1^{m+n} : \mathcal{A}) \;\leq\; S(x_1^m : \mathcal{A}) \cdot S(x_{m+1}^{m+n} : \mathcal{A}).$$

4. A sequence of numbers $\{a_n\}$ is super-additive if for all $m, n \geq 1$ the inequality $a_{m+n} \geq a_m + a_n$ holds. Use the lemma from class to show that if $\{a_n\}$ is super-additive then a_n/n has a limit, and to find the limit.

5. Show that for every $x \geq 0$ one has $\log(1+x) \leq x - x^2/2 + x^3/2$, and conclude that

$$1 + x \;\leq\; \exp\left\{x - \left(\frac{x^2 - x^3}{2}\right)\right\}.$$

Hint: Expand the function $h(v) = \log v$ in a fourth order Taylor series around the point $v = 1$ and consider the remainder term.

6. Let $X \sim \chi^2_k$ have a chi-squared distribution with k degrees of freedom.

(a) Using an identity from a previous homework, or a direct argument, show that if Z is standard normal and $s < 2$ then $\mathbb{E}\exp\{sZ^2\} = (1-2s)^{-1/2}$.

(b) Show that the MGF of X is equal to $\varphi_X(s) = (1-2s)^{-k/2}$.

(c) Use the Chernoff bound and result of Problem 5 above to establish that for $0 \leq \epsilon \leq 1$,

$$P(X \geq (1+\epsilon)k) \leq \exp\left\{-\frac{k}{4}(\epsilon^2 - \epsilon^3)\right\}$$