STOR 655 Homework 10

1. Let $X_1, \ldots, X_n$ be random variables with moment generating functions $\varphi_{X_i}(s) \leq \varphi(s)$ for each $s \geq 0$.

(a) Using the argument in class for Gaussian random variables, show that

$$\mathbb{E}\max(X_1, \ldots, X_n) \;\leq\; \inf_{s>0} \frac{\log n + \log \varphi(s)}{s}.$$

Suppose now that $U_1, \ldots, U_n$ are Gamma(α, β) random variables.

(b) Show that the moment generating function of U_i is $\varphi(s) = (1 - s\,\beta)^{-\alpha}$.

(c) Using the bound from part (a) and an appropriate choice of s, which can be found by inspection, show that

$$\mathbb{E}\max(U_1, \ldots, U_n) \;\leq\; \frac{2\,\beta\,\log n}{1 - n^{-1/\alpha}}.$$

2. Let $X_1, X_2, \ldots, X$ be i.i.d. random variables and let $M_n = \max\{X_1, \ldots, X_n\}$. Show that M_n converges with probability one to the essential supremum $||X||_\infty$ of X.

3. Let $X_1, \ldots, X_n$ be independent standard normal random variables. Here we identify upper and lower bounds for the expectation of $K_n := \max_{1 \leq i \leq n} |X_i|$.

(a) Using the bound from class and the fact that $K_n = \max_i(X_i, -X_i)$ show that $\mathbb{E}K_n \leq (2\,\log 2n)^{1/2}$.

(b) Let $\Phi()$ be the CDF of the standard normal. Show that

$$K_n \;=\; \Phi^{-1}\left(\frac{1}{2} + \frac{1}{2}\max_{1 \leq i \leq n} V_i\right)$$

where $V_1, \ldots, V_n$ are independent Uniform$(0, 1)$ random variables.

(c) Show that $\Phi^{-1}(u)$ is convex on $[1/2, 1)$. Apply Jensen's inequality to the expression in (b) to obtain the bound $\mathbb{E}K_n \geq \Phi^{-1}\left(1 - 1/(2n + 2)\right)$.

(d) Show that $\Phi^{-1}(1 - t^{-1})/(2\,\log t)^{1/2} \to 1$ as $t \to \infty$.

(e) Conclude from (a), (c), and (d) that $\mathbb{E}K_n/(2\,\log n)^{1/2} \to 1$ as $n \to \infty$.

4. *Extreme value theory for the Gaussian.* Let a_n and b_n be the extreme value scaling and centering constants for the maximum M_n of n independent standard Gaussian random variables.

(a) Fix $x \in \mathbb{R}$ and let $x_n = x/a_n + b_n$. Show that $n\,\phi(x_n)/x_n \to e^{-x}$ as n tends to infinity. [In your calculations, identify and pay careful attention to the leading order terms.]

(b) Using the result of part (a) and the standard Gaussian tail bound from an earlier homework, show that $n(1 - \Phi(x_n)) \to e^{-x}$.

(c) Use part (b) and the lemma from lecture to show that as n tends to infinity

$$\mathbb{P}(a_n(M_n - b_n) \leq x) \to G(x) = e^{-e^{-x}}$$

(d) Show that $G(x)$ is the CDF of $-\log V$ where $V \sim \text{Exp}(1)$.

5. Let $U_1, \ldots, U_n$ be independent Uniform$(0, \theta)$ random variables. Find $\mathbb{E}\left[\max_{1 \leq j \leq n} U_j\right]$.

6. Let $\{C_\lambda : \lambda \in \Lambda\}$ be convex sets. Show that the intersection $C = \cap_{\lambda \in \Lambda} C_\lambda$ is convex.

7. Show that the following subsets of $\mathbb{R}^d$ are convex.

a. The emptyset

b. The hyperplane $H = \{x : x^t u = b\}$

c. The halfspace $H_+ = \{x : x^t u > b\}$

d. The Ball $B(x_0, r) = \{x : ||x - x_0|| \leq r\}$

8. Show that if $f_1, \ldots, f_k$ are convex functions defined on the same set, and $w_1, \ldots, w_k$ are non-negative, then $f = \sum_{j=1}^k w_j\, f_j$ is convex.

9. Let $\{f_\lambda : \lambda \in \Lambda\}$ be convex functions defined on a common set C. Show that the supremum $f = \sup_{\lambda \in \Lambda} f_\lambda$ is convex.

10. Let f be a convex function on an open interval $I \subseteq \mathbb{R}$ and let $a < b < c$ be in I.

(a) Show that
$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a} \leq \frac{f(c) - f(b)}{c - b}.$$
(Hint: express b as a convex combination of a and c and then apply the definition of convexity.)

(b) Draw a picture illustrating this result. Interpret the result in terms of the slopes of chords of the function f.

Now let $u < v$ be points in I. For each $x \in I$ let $g(x)$ be the height of the line determined by $(u, f(u))$ and $(v, f(v))$.

(c) Write down a formal expression for $g(x)$ as a function of x.

(d) Use part (a) applied to appropriate points a, b, c to show that $f(x) \leq g(x)$ for $u \leq x \leq v$.

(e) Use part (a) applied to appropriate points a, b, c to show that $f(x) \geq g(x)$ for $x < u$ and $x > v$.