STOR 655 Homework 6

1. Show that $f : \mathbb{R}^d \to \mathbb{R}$ is upper semicontinuous if and only if the super-level sets $\{x : f(x) \geq \alpha\}$ are closed for every $\alpha \in \mathbb{R}$.

2. Show that if $f_1, f_2, \ldots : \mathbb{R}^d \to \mathbb{R}$ are u.s.c. then so is $g(x) = \inf_n f_n(x)$.

3. Let X be a random variable with a finite variance and let $Y = \min(X, c)$ for some constant c. Show that the variance of Y exists and is less that or equal the variance of X. [Hint: By considering $Y - c$, show that the assertion is valid for every c if it is valid for $c = 0$. For the case $c = 0$, express X in terms of Y and $Z = \max(X, 0)$, and then consider the covariance of Y and Z.]

4. Let $\text{Bin}(n, p)$ denote the binomial distribution with parameters $n \geq 1$ and $p \in [0, 1]$. Show that for each $1 \leq k \leq n$ and each $p \in [0, 1]$ that the following identity holds:

$$P(\text{Bin}(n, p) \geq k) \;=\; \frac{n!}{(k-1)!(n-k)!} \int_0^p u^{k-1}(1-u)^{n-k} du$$

Hint: Fix $1 \leq k \leq n$. Let $f(p)$ and $g(p)$ be, respectively, the left- and right-hand sides of the equation. Show that f, g are equal when $p = 0$. Then show that $f'(p) = g'(p)$ for each p. To do this, write $f(p)$ as a sum, differentiate each summand, and then note that terms in successive summands cancel.

5. Let $X_1, \ldots, X_n$ be i.i.d. $\text{Exp}(1)$ random variables.

(a) Write down the joint density of $X = (X_1, \ldots, X_n)$ using indicator functions to capture the fact that the variables X_i are positive.

(b) For $1 \leq k \leq n$ define the random variable $Y_k = X_1 + \cdots + X_k$. Use the general change of variables formula to find the density of $Y = (Y_1, \ldots, Y_n)$.