STOR 655 Homework 5

1. Let $W_n \sim \chi^2_n$ be a chi-squared random variable with n degrees of freedom, and let $\chi^2_{n,\alpha}$ be the upper $1-\alpha$ percentile of the χ^2_n distribution.

   (a) Following the arguments in class, find $\mathbb{E}W_n$ and $\mathrm{Var}(W_n)$, and show that

   $$\frac{W_n - \mathbb{E}W_n}{\mathrm{Var}(W_n)^{1/2}} \Rightarrow \mathcal{N}(0,1)$$

   (b) Use part (a) of the problem to establish the (non-stochastic) relation

   $$\frac{\chi^2_{n,\alpha} - n}{\sqrt{n}} \to \sqrt{2} z_\alpha$$

   where z_α is the $1-\alpha$ upper percentile of the standard normal. Hint: If the desired result fails to hold, then there is a subsequence $\{n_k\}$ along which the centered and scaled percentiles converge to a number greater than, or less than, $\sqrt{2}z_\alpha$. Use this to get a contradiction.

2. Establish the following linear algebra facts from class. Let $A, B \in \mathbb{R}^{n\times n}$.

   (a) If A is a projection matrix then all of its eigenvalues are zero or one.

   (b) If A is a projection matrix then $\mathrm{rank}(A) = \mathrm{tr}(A)$.

   (c) If A is a symmetric projection matrix then $A\,v$ is orthogonal to $v - A\,v$ for every v.

   (d) $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$

3. Let $X_1, X_2, \ldots \in \mathbb{R}^d$ be i.i.d. random vectors with $\mathbb{E}X_i = \mu$ and $\mathrm{Var}(X_i) > 0$. Let

$$T_n^2 = (n-1)(\overline{X}_n - \mu)^t S_n^{-1} (\overline{X}_n - \mu)$$

be Hotelling's T^2 statistic, where $S_n = n^{-1}\sum_{i=1}^n (X_i - \overline{X}_n)(X_i - \overline{X}_n)^t$. Show as carefully as you can that $T_n^2 \Rightarrow \chi^2_d$.

4. Let the sample correlation coefficient r_n of a bivariate data set be defined as in class. Show that $1 \le r_n \le 1$.

5. Show that if $Q \in \mathbb{R}^{n \times n}$ is orthogonal then $||Qx|| = ||x||$ for every x. What does this tell you about the real eigenvalues of Q? Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Use the spectral decomposition of A to show that

$$\sup_{x:\, x^T x = 1} x^T A x \;=\; \lambda_n$$

where λ_n is the largest eigenvalue of A. Deduce from this that

$$\sup_{x \neq 0} \frac{x^T A x}{x^T x} \;=\; \lambda_n.$$

Find a vector for which the inequality is satisfied with equality.